A New Scalar Transition Finite Element for Accurate Analysis of Waveguides with Field Singularities

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***Abstract*—When the scalar Helmholtz equation is solved by Finite Element Method, a slow rate of convergence and inaccurate results can be found if waveguides containing field singularities are analyzed. By using singular elements around the singularity, an important improvement is attained, but when the size of the singular elements is too small, the correct modeling of the fields is lost. In this paper we introduce a new family of transition finite elements that represents a behavior of the axial field component of the form $O(r^{\lambda})$, with any $0 < \lambda < 1$, for singular points placed outside the element. By employing these transition elements adjacent to singular elements, the loss of the singular modeling of the fields is avoided.**

I. Introduction

IN THE analysis of homogeneous waveguides, it is common to find geometries with sharp metal edges in which transversal fields can become infinite.

Several finite element implementations can be found in the literature that cope with the singular behavior of fields. One possibility is to employ edge elements [1],[2], although the standard trial functions cannot model the field singularity in the sharp metal edges adequately. Other authors have proposed to expand the trial function space with singular functions, associating each with a nodeless variable [3]; this choice has the drawback of increasing the bandwidth of the matrix [4].

When the equation to be solved is scalar, as it happens in the quasi-TEM analysis of transmission lines and in the analysis of homogeneous waveguides, a simple and advantageous choice is to employ scalar singular elements (SE), such as those proposed in [5]–[7]. This way, the accuracy and speed of convergence of the method are increased and a good approximation of transversal fields in the singular zone is reached.

Nevertheless, the correct choice of the size of singular elements is a subject that has not been explored enough. It can generally be expected that by using refinements of the mesh around the points where the singularity is produced, the results are improved. However, if the singular elements are too small, the singular modeling of the fields can be lost, raising the error sharply [8].

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To avoid this error, some authors have proposed the use of special elements [8], [9], named transition elements (TE), which are located between the singular elements and the remaining standard elements (StE).

However, there are not any transition elements in the applicable literature that can be employed when the order of singularity of the fields is an irrational number, as is found in the planar transmission lines.

In this paper, a new family of transition finite elements is developed that can represent a singularity with any order when this singularity is placed on a point external to the element. Transition elements avoid the loss of singular modeling of the fields when they are placed adjacent to singular elements such as those presented in [5] and [7].

II. Size of Singular Zone

The appropiated functional for TEM or quasi-TEM modes in transmission lines

$$F(u) = 1/2 \int_s \varepsilon(\nabla_t u)^2 ds \tag{1}$$

or the corresponding one for TE and TM modes in homogeneous waveguides

$$F(u) = 1/2 \int_s [(\nabla_t u)^2 - K_c^2 u^2]\, ds \tag{2}$$

where K_c is the cutoff wavenumber can be easily solved by the FEM. However, when the analyzed structure includes field singularities, the accuracy and speed of convergence of the method are increased by using a refinement of the mesh in the neighbourhood of the singularity, surrounding it with SE and employing a rough mesh in areas of the section where smooth variations of the fields are expected.

In practice, it can be noted [10] that with a fixed number of degrees of freedom, the accuracy of the solution substantially depends on a parameter denominated size or diameter of the singular zone s (Fig. 1). In other words, it depends on the size of the region surrounding the singular point meshed with SE. The accuracy tends to be improved when s is decreasing but, when the SE are too small, the modeling of singular term is partially lost. Then, a new source of error appears because the adjacent elements to singular zone (StE) have to represent a part of singular term. Obviously, there will be an optimum value of s for each problem and analyzed mode.

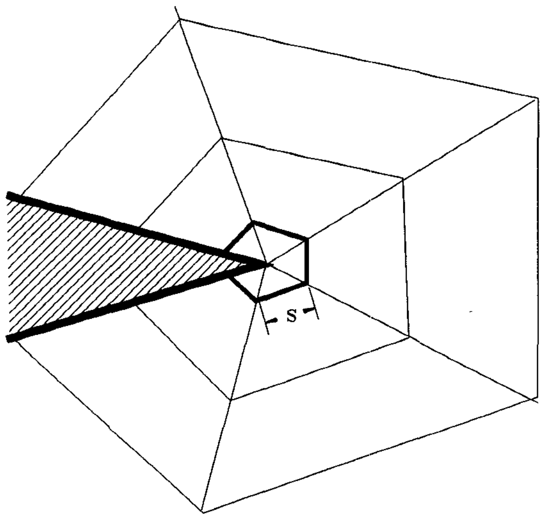

Fig. 1. Mesh of the singular zone.

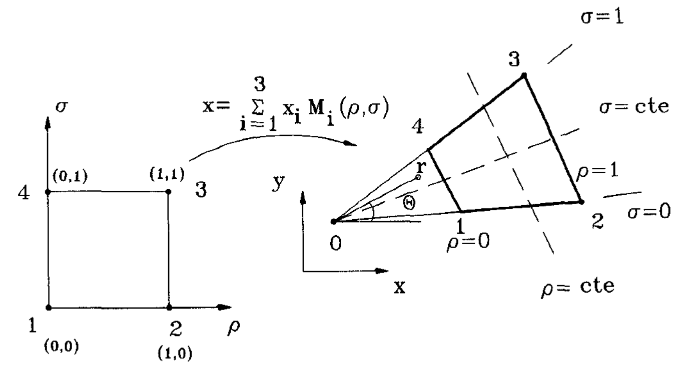

Fig. 2. Mapping of new transition element.

A better approximation of the fields, and therefore of the eigenvalue, can be expected if the adjacent StE to singular zone are replaced with other elements suitable to model the singular behavior of the fields in a point external to the element. This way, though the SE are very small, the term r^λ is adequately modelled over the whole area of interest. If the size of SE is correctly chosen, the advantages obtained by employing these TE are reduced, but the correct size is unknown *a priori*.

This paper will show that meshes using a mixture of SE, TE, and StE, give an accuracy in the results that is greater or equal to that obtained when we use only SE and StE, and this is true for any size of singular and transition zones that we choose.

III. New Family of Transition Elements

The new transition elements are cuadrilateral elements that can represent a behavior of the axial field component of the form $O(r^\lambda)$ with any $0 < \lambda < 1$, located on a point external to the element. The unknown function is approximated with Lagrange polynomials, and the singularity is introduced in terms of the geometrical transformation (Fig. 2).

This transformation is given by means of the following expression:

$$t - t_0 = \left[\frac{1+\rho(\alpha_0^{1/m}-1)}{\alpha_0^{1/m}}\right]^m [(t_2 - t_0) + \sigma(t_3 - t_2)] \quad (3)$$
$$t = x, y$$

where $t = x, y$ are the coordinates in real space; $t_0 = x_0, y_0$ are the coordinates of the singular point; $t_i = x_i, y_i$, with $i = 1, 4$ are the vertex coordinates of the element; ρ y σ are the coordinates in local space; m is a parameter; and α_0 is defined as

$$\alpha_0 = \frac{dO3}{dO4} = \frac{dO2}{dO1} \quad (4)$$

with dOi the distance from the point O (singular point) to the vertex i. It is sufficient to know the coordinates of three of the vertex and the parameter α_0 to define the new element.

The Jacobian of the geometrical transformation is

$$j = m\frac{(\alpha_0^{1/m}-1)}{\alpha_0^{1/m}} J_0 \left[\frac{1+\rho(\alpha_0^{1/m}-1)}{\alpha_0^{1/m}}\right]^{2m-1} \quad (5)$$

where J_0 is twice the surface of triangle $O23$. The Jacobian vanishes if

$$\rho = -\frac{1}{\alpha_0^{1/m}-1} \Longrightarrow t = t_0 \quad (6)$$

and, therefore, the transformation is not invertible in this point.

It can be easily shown [10] that along a straight line through the singular point, ρ it is proportional to

$$r^{1/m}. \quad (7)$$

On the other hand, if we employ a second order approximation for the unknown function, u, the shape functions have these terms

$$N_i \to (\rho, \sigma, \rho^2, \sigma^2, \rho\,\sigma, \text{ constants}) \quad (8)$$

and, in the neighborhood of the singular point, the function u and its gradient exhibit a behavior

$$u \to O(r^{2/m}, r^{1/m}, \text{ constants})$$
$$\nabla u \to O\left(r^{\frac{2-m}{m}}, r^{\frac{1-m}{m}}, \text{ constants}\right). \quad (9)$$

When r is small, the principal term is

$$r^{\frac{1-m}{m}} \quad (10)$$

which represents the singular behavior of the field.

If the principal term to be modelled is of the form $O(r^\lambda)$, it would be sufficient to take $m = 1/\lambda$ in the geometrical transformation to represent the singularity adequately. An analogous discussion could be applied when higher-order shape functions are used, and it is easy to check how the proposed element is compatible with any order StE adjacent to the edge 23.

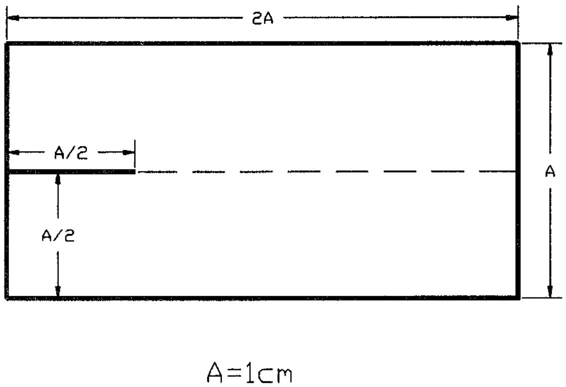

Fig. 3. Rectangular vaned waveguide (A = 1 cm).

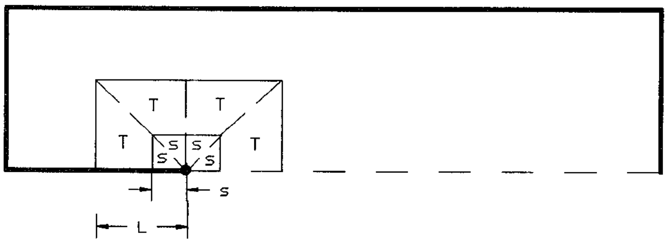

Fig. 4. Mixed mesh of the rectangular vaned waveguide with the singular and transition zones. s = Size of singular zone. L = Size of singular + transition zone.

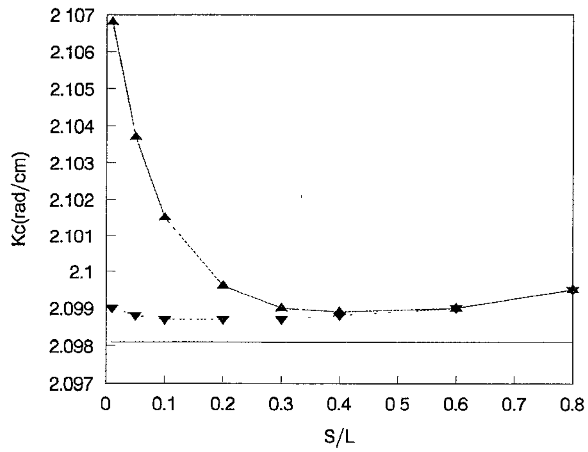

Fig. 5. K_c of the first singular TE mode in the rectangular vaned waveguide versus the relation between the size of the singular zone (s) and the size of the singular + transition zone (L), with L = 0.25 cm. $\triangle$: singular and standard elements. ∇: singular, transition and standard elements. ——: value of convergence K_c = 2.0981 rad/cm (2047 nodes).

IV. Numerical Results

In order to evaluate the improvement obtained by using TE, the first singular TE mode in a rectangular vaned waveguide, as shown in Fig. 3, has been studied. This structure has an 180° wedge with a transversal field singularity on the edge of the form $O(r^{-1/2})$. The eigenvalue, K_c =2.0981 rad/cm, chosen as value of convergence, has been obtained using SE and a uniform mesh of 2047 nodes.

In this case, a computationally more efficient mesh has been employed, composed of singular triangular elements as proposed in [5], standard cuadrilateral elements, and a layer of TE between both of them. This situation is shown in Fig. 4, which represents a mesh of the structure with 4 SE, 4 TE, 14 StE, and 70 degrees of freedom.

In Fig. 5. the eigenvalue K_c versus s/L is shown, with L as a parameter, where s is the size of the singular zone and L is the size of the singular + transition zone. The results obtained by using the mesh with StE. + T.E. + S.E. are compared with results obtained with the same mesh but replacing TE for StE. It can be observed in this case that when SE are too small (i.e., s/L =0.01), the error increases sharply. This can be explained because the StE must represent a part of singular term. This source of error disappears when we replace the StE with TE. It can also be noticed that, when SE are relatively large (i.e., s/L =0.8), the improvement obtained when we substitute StE for TE is negligible because the singular term is correctly approximated.

It is known that by using StE the results are affected by the employment of adjacent elements of very different size that degrades the FEM aproximation. In this case, as the same meshes were employed for computing both curves in the figure, the preceding effect would have similar influence in both results.

It has been checked that the described behavior holds for different values of L and λ and, as a consequence, the problem of the correct choice of SE size is minimized. In conclusion, it can be assured that the incorporation of a layer of TE between SE and StE produces results that always are better or equal to those obtained when TE are not employed.

The loss of singular modeling when SE are very small is clarified in Fig. 6(a) and (b). There, the radial component of the gradient of u (transversal field) in the neighborhood of the singular point, following the $\Theta = \pi/4$ direction, has been represented. In Fig. 6(a), this field is shown when TE are employed. An excellent agreement with the expected theoretical value $O(r^{-1/2})$ is confirmed. On the contrary, in Fig. 6(b) the loss of this agreement can be seen. This effect is produced because of the replacement of TE by StE, which cannot represent the singular variation of the field.

Finally, in Fig. 7, the advantage of using the StE. + T.E. + S.E. mesh, as opposed to only the StE mesh, is shown. It can be seen that the solution obtained in the first case is practically independent of the size of the SE. The value of convergence, K_c = 2.0981 rad/cm, was obtained by using the SE and a uniform mesh of 2047 nodes.

Similar conclusions were obtained by analyzing structures containing singularities of different orders to those previously described. Results on the first TE mode of a single ridge waveguide, containing a singularity of the form $O(r^{-1/3})$, can be found in Fig. 8. A mesh with 6 SE, 6 TE, 6 StE, and a total of 58 degrees of freedom have been employed. In the figure the mesh and the eigenvalue versus the relation of the sizes s/L with and without the employment of TE, are shown. Once more, value of convergence (K_c = 2.2497 rad/cm) was computed with the SE and a uniform mesh of 1541 nodes.

Again, when the SE are too small, the error can be significant if the TE are not employed.

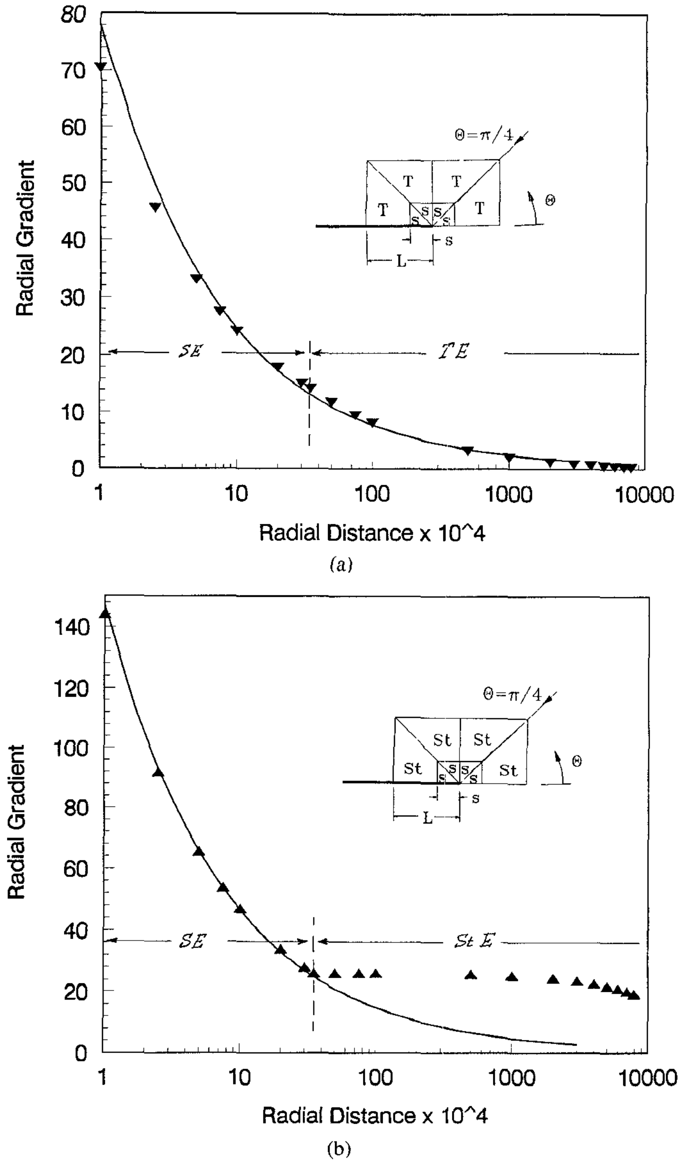

Fig. 6. (a) Radial gradient versus the distance to singular point, following the $\Theta = \pi/4$ direction, with transition elements. $s/L = 0.01$. ∇: singular + transition elements. ——: theoretical variation. (b) Radial gradient versus the distance to singular point, following the $\Theta = \pi/4$ direction, without transition elements. $s/L = 0.01$. Δ: singular + standard elements. ——: theoretical variation.

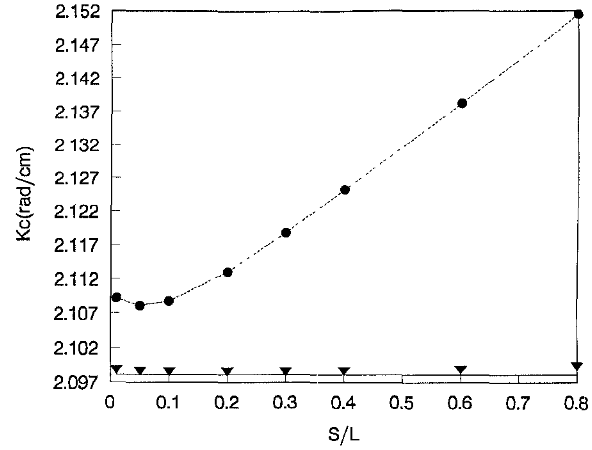

Fig. 7. K_c of the first singular TE mode in the rectangular vaned waveguide versus the relation between the size of the singular zone (s) and the size of the singular + transition zone (L), with $L = 0.25$ cm, in a mesh with 85 nodes by using singular, transition, and standard elements (∇) and substituting singular and transition elements for standard elements (•). ——: value of convergence $K_c = 2.0981$ rad/cm (2047 nodes).

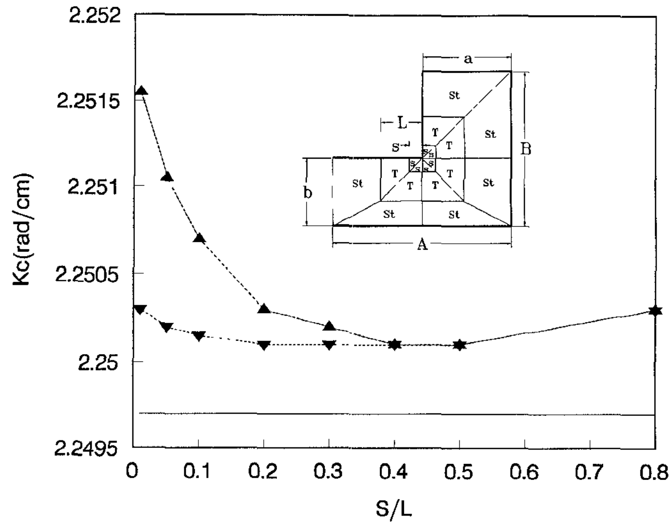

Fig. 8. K_c of the first TE mode, in the single ridge waveguide, versus the relation between the size of the singular zone (s) and the size of the singular + transition zone (L), with $L = 0.125$ cm, in a mesh with 58 degrees of freedom. $A = B = 2a = 2b = 0.5$ cm. Δ: singular and standard elements. ∇: singular, transition and standard elements. ——: value of convergence $K_c = 2.2497$ rad/cm (1541 nodes).

V. Conclusion

It has been shown that the accuracy improvement obtained by using singular elements (SE) in the FEM resolution of the scalar Helmholtz equation, when transversal field singularities are present, is lost if the size of singular elements is very small. This is because a part of singular term is not adequately modelled.

A new family of transition finite elements (TE) has been developed suitable to represent a behavior of the axial field component of the form $O(r^{\lambda})$ with any $0 < \lambda < 1$ when the singular point is external to the element.

These new elements are compatible with any order standard elements (StE) and they provide improvement in the accuracy of the solution of waveguides with field singularities. The improvement is practically independent of the size of the singular elements used.

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